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PLASTIC FLOW OF CONE-SHAPED BODIES*

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Flows of an ideal rigidly-plastic incompressible medium, in the shape of a cone-shaped body, are considered for different external effects. The problem of axisymmetric flow reduces to a system of two ordinary differential equations whose solutions describe the limit state of conical tubes subjected to uniformly distributed annular tangential forces, normal and annular tangential forces, normal and longitudinal tangential forces on the inner and outer surfaces. The combined bending and tension of a conical sheet and the flow of plastic mass between two-dimensionally rough conical surfaces which are approaching exponentially in the annular coordinate are investigated.

The axisymmetric radial flow of a plastic mass in convergent channels in the shape of a circular cone is investigated in /1,2/. The problem of the limit state of a conical tube under uniform internal and external pressure is solved in /3/, while the solution of the corresponding elastic-plastic problem is constructed in /1/. The flow of plastic material between conical surfaces taken rough in the annular direction is investigated in /4,5/ for constant transverse velocities in this same direction.

1. Fundamental Equations. The relationships of the theory of an ideal rigidly-plastic flow in spherical coordinates have the following form in the usual notation:
Equilibrium differential equations

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\varphi}}{\partial\varphi}+\frac{1}{r}(2\sigma_r-\sigma_\theta-\sigma_\varphi+\tau_{r\theta}\operatorname{ctg}\theta)=0 \tag{1.1}$$
$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\varphi}}{\partial\varphi}+\frac{1}{r}[(\sigma_\theta-\sigma_\varphi)\operatorname{ctg}\theta+3\tau_{r\theta}]=0$$
$$\frac{\partial\tau_{r\varphi}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta\varphi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\sigma_\varphi}{\partial\varphi}+\frac{1}{r}(2\tau_{\theta\varphi}\operatorname{ctg}\theta+3\tau_{r\varphi})=0$$

Relationships between strain rate, stress, and displacement velocity comments

$$\varepsilon_{ij}=\Lambda(\sigma_{ij}-\delta_{ij}\sigma),\quad \varepsilon_r=\frac{\partial u}{\partial r},\quad \varepsilon_\theta=\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta} \tag{1.2}$$
$$\varepsilon_\varphi=\frac{1}{r\sin\theta}\frac{\partial w}{\partial\varphi}+\frac{u}{r}+\frac{v}{r}\operatorname{ctg}\theta,\quad 2\gamma_{r\varphi}=\frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi}+\frac{\partial w}{\partial r}-\frac{w}{r}$$
$$2\gamma_{r\theta}=\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta},\quad 2\gamma_{\theta\varphi}=\frac{1}{r}\frac{\partial w}{\partial\theta}-\frac{w}{r}\operatorname{ctg}\theta+\frac{1}{r\sin\theta}\frac{\partial v}{\partial\varphi}$$

Huber-Mises plasticity condition

$$(\sigma_r-\sigma_\theta)^2+(\sigma_\theta-\sigma_\varphi)^2+(\sigma_\varphi-\sigma_r)^2+6(\tau_{r\theta}^2+\tau_{\theta\varphi}^2+\tau_{r\varphi}^2)=6 \tag{1.3}$$

Here and henceforth, the stress components are referred to the plastic constant k. It is convenient to represent the stress components in the form

$$\sigma_r=\sigma_\theta+\frac{1}{\Omega_0}(\varepsilon_r-\varepsilon_\theta),\quad \sigma_\varphi=\sigma_\theta-\frac{1}{\Omega_0}(\varepsilon_r+2\varepsilon_\theta) \tag{1.4}$$
$$\tau_{ij}=\frac{1}{\Omega_0}\gamma_{ij},\quad \Omega_0=(\varepsilon_r^2+\varepsilon_r\varepsilon_\theta+\varepsilon_\theta^2+\gamma_{r\theta}^2+\gamma_{\theta\varphi}^2+\gamma_{r\varphi}^2)^{1/2}$$

In a certain class of axisymmetric flows the stress and displacement velocity components can be expressed in terms of the unknown functions $f(\theta),\psi(\theta)$ and arbitrary constants in the following form:

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$$\sigma_r=\sigma_\theta+\frac{2}{\Omega}[(2\lambda+1)f'+(\lambda-1)f\operatorname{ctg}\theta], \tag{1.5}$$
$$\sigma_\varphi=\sigma_\theta+\frac{2(\lambda+2)}{\Omega}(f'-f\operatorname{ctg}\theta)$$
$$\tau_{r\theta}=\frac{1}{\Omega}[g'+(1-\lambda)(2+\lambda)f], \quad \tau_{r\varphi}=\frac{(\lambda-1)(\lambda+2)}{\Omega}\psi\sin\theta$$
$$\tau_{\theta\varphi}=\frac{\lambda+2}{\Omega}\psi'\sin\theta, \quad \Omega=\{4\lambda^2g^2-4\lambda gh+4h^2+$$
$$[g'+(1-\lambda)(2+\lambda)f]^2+(\lambda+2)^2\sin^2\theta[\psi'^2+(\lambda-1)^2\psi^2]\}^{1/2}$$
$$g=f'+f\operatorname{ctg}\theta, \quad h=(\lambda+1)f'-f\operatorname{ctg}\theta, \quad \lambda=\text{const}$$
$$u=r^\lambda g, \quad v=-(\lambda+2)r^\lambda f, \quad w=(\lambda+2)r^\lambda\psi\sin\theta+Cr\sin\theta$$

Here and henceforth the letters of the upper case of Roman alphabet denote arbitrary constants.

The expression

$$\sigma_\theta=-H+M\ln r-3\int\tau_{r\theta}\,d\theta+2(\lambda+2)\int(f'-f\operatorname{ctg}\theta)\operatorname{ctg}\theta\frac{d\theta}{\Omega} \tag{1.6}$$

and the system of two ordinary nonlinear differential equations in f and ψ

$$\left\{\frac{\sin\theta}{\Omega}[f''+(f\operatorname{ctg}\theta)'+(1-\lambda)(2+\lambda)f]\right\}'+ \tag{1.7}$$
$$\frac{6\lambda}{\Omega}(f\sin\theta)'+M\sin\theta=0$$
$$\left(\frac{\sin^3\theta}{\Omega}\psi'\right)'-3(1-\lambda)\frac{\sin^3\theta}{\Omega}\psi=0$$

follow from the equilibrium equations (1.1) and the relations (1.5).

The boundary conditions for the system (1.7) are determined by specific flow conditions. For $\lambda=-2$, $\psi=C=0$ we have the radial flow case considered in /1,2/.

2. Torsion of a conical tube by annular tangential forces acting on the side surfaces. We assume that tangential distributed loads

$$\tau_{\theta\varphi}=q_1 \text{ as } \theta=\alpha, \quad \tau_{\theta\varphi}=q_2 \text{ as } \theta=\beta \tag{2.1}$$

are applied to the inner and outer side surfaces of a long conical tube.

Upon substitution of $f(\theta)\equiv 0, M=H=0$ into (1.5)–(1.7), we obtain $\sigma_r=\sigma_\theta=\sigma_\varphi=\tau_{r\theta}=0$, $u=v=0$. The second equation in (1.7) can, if the subscripts on $\tau_{\theta\varphi}$ are removed, be written in the form

$$\tau'+2\tau\operatorname{ctg}\theta+3\sqrt{1-\tau^2}=0 \tag{2.2}$$
$$\tau=\psi'[\psi'^2+(\lambda-1)^2\psi^2]^{-1/2}, \quad \lambda\neq 1 \tag{2.3}$$

The solution of equation (2.2) can be obtained in quadratures. By introducing the new function $Z(x)$

$$\tau=\frac{Z(x)}{\sqrt{1+Z^2(x)}}, \quad x=\operatorname{ctg}\theta$$

and then the function

$$Z=\frac{x-y}{1+xy}, \quad y=\frac{x\sqrt{1-\tau^2}-\tau}{\sqrt{1-\tau^2}+x\tau} \tag{2.4}$$

equation (2.2) reduces to a linear differential equation in x whose solution has the form

$$2x(1+y^2)^{1/4}+\int(1+y^2)^{-3/4}\,dy=\text{const} \tag{2.5}$$

The quadrature obtained in expressed in terms of an elliptic integral of the first kind. In the long run, by taking account of the first boundary condition from (2.1), we obtain

$$\frac{\sqrt{2}\operatorname{ctg}\theta}{T(\tau,\theta)}+F\left[\arccos T(\tau,\theta),\frac{1}{\sqrt{2}}\right]=\frac{\sqrt{2}\operatorname{ctg}\alpha}{T(q_1,\alpha)}+F\left[\arccos T(q_1,\alpha),\frac{1}{\sqrt{2}}\right] \tag{2.6}$$

where we used the notation

$$T(x, y) = (x \cos y + \sqrt{1 - x^2} \sin y)^{1/2}$$

Utilization of the second boundary condition in (2.1) yields the relationship

$$\frac{\sqrt{2}\operatorname{ctg}\beta}{T(q_2, \beta)} + F\left[\operatorname{arc\,cos} T(q_2, \beta), \frac{1}{\sqrt{2}}\right] = \frac{\sqrt{2}\operatorname{ctg}\alpha}{T(q_1, \alpha)} + F\left[\operatorname{arc\,cos} T(q_1, \alpha), \frac{1}{\sqrt{2}}\right] \tag{2.7}$$

The relationship (2.7) imposes a connection between the values of q_1 and q_2 which should be conserved in the limit state of the conical tube.

From (2.3) we determine $\psi(\theta)$

$$\psi = D \exp\left[(\lambda - 1)\int_{\alpha}^{\theta} \frac{\tau\, d\theta}{\sqrt{1 - \tau^2}}\right] \tag{2.8}$$

3. Conical tube subjected to the combined action of normal and annular tangential forces. We assume that distributed normal and tangential forces

$$\sigma_\theta = -p_1,\ \tau_{\theta\varphi} = q_1 \text{ as } \theta = \alpha;\ \sigma_\theta = -p_2,\ \tau_{\theta\varphi} = q_2 \quad \text{as} \quad \theta = \beta \tag{3.1}$$

act on the inner and outer surfaces of a long conical tube.

If we take $f = E/\sin\theta$, $\lambda = 1$, $M = 0$ in the relationships (1.5)–(1.7), then the first equation in (1.7) is satisfied identically, and upon taking account of the boundary condition on the inner surface we will obtain from the second (we omit the subscripts on the $\tau_{\theta\varphi}$)

$$\tau = \psi' \sin^3\theta\,(4E^2\cos^2\theta + \psi'^2\sin^6\theta)^{-1/2}, \quad \tau = q_1 \frac{\sin^2\alpha}{\sin^2\theta} \tag{3.2}$$

By hence determining ψ', substituting in the normal stresses (1.5) and (1.6), and taking account of the boundary conditions on the inner surfaces, we have

$$\begin{gathered}\sigma_r = \sigma_\theta + \sqrt{1 - \tau^2},\quad \sigma_\varphi = \sigma_\theta + 2\sqrt{1 - \tau^2}\\ \sigma_\theta = -p_1 + T(\tau, \sin\theta) - T(q_1, \sin\alpha)\\ T(x, y) = \ln[1 + \sqrt{1 - x^2}] - \sqrt{1 - x^2} + 2\ln y\end{gathered} \tag{3.3}$$

The conditions on the outer surface govern the following connection between the parameters p_i, q_i, which should be conserved in a definite state of the conical tube

$$p_1 - p_2 = T(q_2, \sin\beta) - T(q_1, \sin\alpha),\quad q_2\sin^2\beta = q_1\sin^2\alpha \tag{3.4}$$

There follows from (3.2)

$$\psi = \frac{E}{q_1\sin^2\alpha}[\sqrt{1 - \tau^2} - \sqrt{1 - q_1^2}] + \text{const} \tag{3.5}$$

If the values of f are taken into account, the displacement velocities will be

$$u = 0,\quad v = -\frac{3Er}{\sin\theta},\quad w = \frac{3Er\sin\theta}{q_1\sin^2\alpha}[\sqrt{1 - \tau^2} - \sqrt{1 - q_1^2}] + Gr\sin\theta \tag{3.6}$$

Formulas /3/, corresponding to the limit state of a conical tube for internal and external pressure are obtained from the preceding formulas for $q_1 = 0$.

Upon going over to cylindrical coordinates and fixing $r\sin\theta$, the formulas for the stress follow from the (3.3) and (3.4) obtained, as $r \to \infty$, $\theta \to 0$:

$$\begin{gathered}\sigma_r = -p_1 + T(\tau, r) - T(q_1, a)\\ \sigma_\theta = \sigma_r - 2\sqrt{1 - \tau^2},\quad \tau_{r\theta} = \tau = q_1\frac{a^2}{r^2}\end{gathered} \tag{3.7}$$

while relationships for the limit state are obtained from (3.4)

$$p_1 - p_2 = T(q_2, b) - T(q_1, a),\quad q_2 b^2 = q_1 a^2 \tag{3.8}$$

The formulas (3.7) and (3.8) determine the ultimate state of stress of a cylindrical tube in cylindrical coordinates, where a and b here and henceforth denote the inner and outer radii, respectively.

The expressions (3.7) for the stress agreee with the Nadai formulas /6/ referring to the stresses in the plastic domain surrounding a circular cavity in the infinite plane.

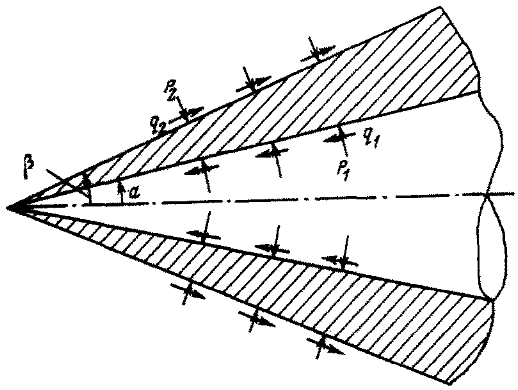

Fig.1

4. Conical tube subjected to the combined action of normal and longitudinal tangential forces (Fig.1). We consider the limit state of a long conical tube when normal pressures $-p_1$, $-p_2$ and tangentially distributed forces q_1, q_2 parallel to the generatrix, respectively, act on the inner and outer side surfaces.

Assuming $\psi(\theta) \equiv 0$, $\lambda = M = 0$ in (1.5)–(1.7), and omitting the subscripts on the $\tau_{r\theta}$, we obtain

$$\tau = q_1 \frac{\sin\alpha}{\sin\theta}, \quad \Omega = \frac{2\nu(f' - f\operatorname{ctg}\theta)}{\sqrt{1-\tau^2}}, \quad \nu = \operatorname{sign}(f' - f\operatorname{ctg}\theta) \tag{4.1}$$

Substituting these expressions in (1.5) and (1.6), and taking account of the condition on the inner surface, we find

$$\begin{gathered}\sigma_r = \sigma_\theta + \sqrt{1-\tau^2}, \quad \sigma_\varphi = \sigma_\theta + 2\sqrt{1-\tau^2} \\ \sigma_\theta = -p_1 + T(\tau, \theta) - T(q_1, \alpha), \; \tau_{r\varphi} = \tau_{\theta\varphi} = 0 \\ T(x, y) = 2\ln[1 + \sqrt{1-x^2}] - 2\sqrt{1-x^2} + 2\ln\sin y - \\ 3q_1\sin\alpha\ln\operatorname{tg} y/2\end{gathered} \tag{4.2}$$

Satisfying the conditions on the outer surface $\theta = \beta$ we arrive at relations determining the limit state of the conical tube in terms of loads

$$p_1 - p_2 = T(q_2, \beta) - T(q_1, \alpha); \; q_2\sin^2\beta = q_1\sin^2\alpha \tag{4.3}$$

The displacement velocities can be taken in the form

$$u = f' + f\operatorname{ctg}\theta, \; v = -2f, \; w = 0 \tag{4.4}$$

Comparing the formula for the tangential stress from (1.5) with the expressions (4.1), we arrive at a linear differential equation in f

$$f'' + \left(\operatorname{ctg}\theta - \frac{2\tau}{\sqrt{1-\tau^2}}\right)f' + \left(2 - \frac{1}{\sin^2\theta} + \frac{2\tau}{\sqrt{1-\tau^2}}\operatorname{ctg}\theta\right)f = 0 \tag{4.5}$$

whose general solution will be

$$f = C_1\sin\theta + C_2\sin\theta\int_\alpha^\theta \frac{1}{\sin^3\theta}\exp\left[2\int_\alpha^\theta \frac{\tau\, d\theta}{\sqrt{1-\tau^2}}\right]d\theta \tag{4.6}$$

The formula /3/

$$p_1 - p_2 = 2\ln\frac{\sin\beta}{\sin\alpha}$$

follows from (4.3) for $q_1 = q_2 = 0$.

The solution in /3/ corresponds to $\lambda = 1$, $\psi = 0$, etc. in the case under consideration, while according to /3/ the corresponding velocity field has the form $u = w = 0$, $v = Nr/\sin\theta$. The displacement velocity field corresponding to the case under consideration here $\lambda = 0$, $\psi = 0$, etc., is determined from (4.4), where

$$f = C_1 \sin\theta + C_2 \operatorname{ctg}\theta + \frac{C_2}{4} \sin\theta \ln\frac{1+\cos\theta}{1-\cos\theta} \tag{4.7}$$

Going over to cylindrical coordinates, setting $r\sin\theta$ as $r \to \infty$, $\theta \to 0$, we obtain from (4.2) and (4.3)

$$\sigma_r = -p_1 + T_0(\tau, r) - T_0(q_1, \alpha)$$

$$\sigma_\theta = \sigma_r + 2\sqrt{1-\tau^2}, \quad \sigma_z = \sigma_r + \sqrt{1-\tau^2}, \quad \tau_{rz} = \tau = q_1 a/r$$

$$T_0(x, y) = 2\ln[1 + \sqrt{1-x^2}] - 2\sqrt{1-x^2} + 2\ln y$$

In cylindrical coordinates these formulas determine the limit state of stress of a cylindrical tube under the combined action of normal pressures and distributed tangential forces.

5. Bending and tension of a conical sheet (Fig.2). Let a sheet in the form of a sector as a long conical tube be in the limit state under the combined effect of distributed bending moments and tensile forces applied to the axial endface sections. The law of variation of these forces along the generators must be determined.

We seek the displacement velocity field in the form

$$u = 0, \quad v = 3r\left(A \operatorname{ctg}\theta - \frac{B}{\sin\theta} - C\cos\varphi\right)$$

$$w = 3r(A\varphi\sin\theta + C\cos\theta\sin\varphi)$$

Then the strain rate components different from zero will be

$$\varepsilon_\varphi = -\varepsilon_\theta = \frac{3}{\sin^3\theta}(A - B\cos\theta)$$

The corresponding stress components are determined from the equations (1.1) and (1.4)

$$\sigma_r = \sigma_\theta - \nu, \quad \sigma_\varphi = \sigma_\theta - 2\nu, \quad \sigma_\theta = -H - 2\nu\ln\sin\theta, \quad \nu = \operatorname{sign}\varepsilon_\theta$$

It follows from the condition of no loading on the inner surface

$$\sigma_r = \sigma_\theta - 1, \quad \sigma_\varphi = \sigma_\theta - 2, \quad \sigma_\theta = -2\ln\frac{\sin\theta}{\sin\alpha} \quad (\alpha \leqslant \theta \leqslant \gamma)$$

where $\theta = \gamma$ is the neutral surface of the layer. Taking account of the conditions $\theta = \beta$ on the outer surface, we determine

$$\sigma_r = \sigma_\theta + 1, \quad \sigma_\varphi = \sigma_\theta + 2, \quad \sigma_\theta = -2\ln\frac{\sin\beta}{\sin\theta} \quad \gamma \leqslant \theta \leqslant \beta$$

From the condition of continuity of σ_θ on the surface $\theta = \gamma$ we find

$$\sin\gamma = \sqrt{\sin\alpha\sin\beta}$$

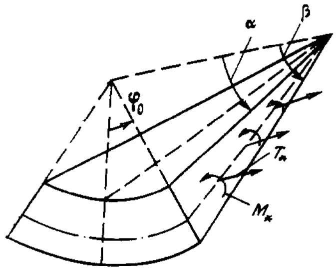

Fig.2

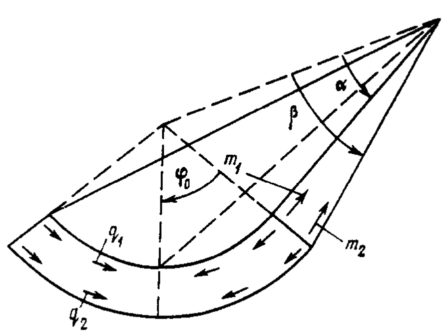

Fig.3

The limit bending moment relative to the axis $\theta = 0$ per unit length will be

$$M_* = r^2 \int_\alpha^\beta \sigma_\varphi \sin\theta \, d\theta = 4r^2 \ln\left(\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin^{-2}\frac{\gamma}{2}\right)$$

Bending moments relative to axes perpendicular to the axis $\theta = 0$ and lying in the axial endface sections $\varphi = \pm\varphi_0$ are zero. Indeed, by expressing σ_φ from the first equilibrium equation (1.1) for the case under consideration, we obtain

$$M^* = r^2 \int_\alpha^\beta \sigma_\varphi \cos\theta \, d\theta = r^2 \sigma_\theta \sin\theta \Big|_\alpha^\beta = 0$$

The ultimate tensile force in the sections $\varphi = \pm\varphi_0$ will be

$$T_* = r \int_\alpha^\beta \sigma_\varphi \, d\theta = r\left[\int_\gamma^\beta \left(1 - \ln\frac{\sin\beta}{\sin\theta}\right) d\theta - \int_\alpha^\gamma \left(1 + \ln\frac{\sin\theta}{\sin\alpha}\right) d\theta\right]$$

Assuming $\varepsilon_\varphi = 0$ on the surface $\theta = \gamma$ and clamping the line $\varphi = 0, \theta = \gamma$, we find

$$\varepsilon_\varphi = -\varepsilon_\theta = \frac{3B}{\sin^2\theta}(\cos\gamma - \cos\theta)$$

$$\frac{v}{3Br} = \sin\gamma\cos\varphi - \frac{1 - \cos\gamma\cos\theta}{\sin\theta}$$

$$u = 0, \quad \frac{w}{3Br} = \varphi\cos\gamma\sin\theta - \sin\gamma\cos\theta\sin\varphi$$

Upon going over to cylindrical coordinates and fixing $r\sin\theta$ as $r \to \infty$, $\theta \to 0$ the Hill solution /7/ is obtained from the formulas obtained for the bending of a cylindrical sheet (in particular, $M_* = (b-a)^2/2$, $T_* = 0$).

6. Flow of a plastic mass between rough conical surfaces (Fig.3). The problem of the limit state of a plastic material, between rigid rough slabs was first investigated by Prandtl in /8/ under plane strain conditions. The theory of the flow of plastic material over rigid surfaces was later developed and extended in /1-7/,9-12,13/.

The problem of the plastic flow between rigid rough conical surfaces is studied in /11/. In contrast to the problem under consideration in which the transverse displacement velocity is considered an exponential function of the azimuthal coordinate, this velocity is assumed constant in the designated coordinate in /11/.

Let us examine the problem of the flow of an incompressible ideally plastic mass between two-dimensional rough conical surfaces as they come towards each other according to the law

$$v = \omega_1 r e^{-\mu|\varphi|} \text{ as } \theta = \alpha, \quad v = -\omega_2 r e^{-\mu|\varphi|} \text{ as } \theta = \beta \tag{6.1}$$

where ω_1, ω_2, μ are given constants. Because of symmetry, we consider the domain $0 \leqslant \varphi \leqslant \varphi_0$. We assume the tangential stresses occurring in the contact surfaces /1,9/ are considerably less than the material yield under the shear and equal, respectively:

$$\tau_{r\theta} = m_1, \quad \tau_{\theta\varphi} = q_1, \quad \theta = \alpha$$
$$\tau_{r\theta} = -m_2, \quad \tau_{\theta\varphi} = -q_2, \quad \theta = \beta \tag{6.2}$$

evidently $m_i^2 + q_i^2 < 1$.

We seek the stress and displacement velocity components in the form

$$\sigma_r = \sigma_\theta + \frac{6}{\Omega}f', \quad \sigma_\varphi = \sigma + \frac{6}{\Omega}(f' - \Phi) \tag{6.3}$$

$$\tau_{r\theta} = \frac{1}{\Omega}(f' + \Phi)', \quad \tau_{\theta\varphi} = \frac{3}{\Omega}\left(\psi'\sin\theta + \frac{\mu f}{\sin\theta}\right)$$

$$\tau_{r\varphi} = -\frac{\mu}{\sin\theta}(f' + \Phi), \quad \Omega = \left[(f' + \Phi)'^2 + 9\left(\psi'\sin\theta + \frac{\mu f}{\Omega\sin\theta}\right)^2 + \left(4 + \frac{\mu^2}{\sin^2\theta}\right)(f' + \Phi)^2 + 4(2f' - \Phi)(f' - 2\Phi)\right]^{1/2}$$

$$u = r(f' + \Phi)e^{-\mu\varphi}, \quad v = -3rfe^{-\mu\varphi}, \quad w = 3r\psi e^{-\mu\varphi}\sin\theta \tag{6.4}$$

where f and ψ are arbitrary functions of θ, while $\Phi = f\operatorname{ctg}\theta + \mu\psi$.

Substituting (6.3) into the equilibrium differential equations (1.1), we arrive at the expression

$$\sigma_\theta = -H + M\ln r - A(\varphi_0 - \varphi) + 6\int_\alpha^\theta (f' - \Phi)\frac{\operatorname{ctg}\theta\, d\theta}{\Omega} - 3\int_\alpha^\theta \tau_{r\theta}\, d\theta \tag{6.5}$$

and the system of two ordinary differential equations with the boundary conditions

$$\left[\frac{\sin\theta}{\Omega}(f + \Phi)\right]' + \frac{6\sin\theta}{\Omega}(f' + \Phi)' + M\sin\theta = 0 \tag{6.6}$$

$$\left[\frac{\sin\theta}{\Omega}(\psi\,\sin^2\theta + \mu f)\right]' - \frac{\mu\sin\theta}{\Omega}(f' + \Phi) + \frac{A}{3}\sin\theta = 0$$

$$f = -\omega_1/3 \quad \text{as} \quad \theta = \alpha, \quad f = \omega_2/3 \quad \text{as} \quad \theta = \beta \tag{6.7}$$

$$\frac{1}{\Omega}(f' + \Phi)' = \begin{cases} m_1 & \theta = \alpha \\ -m_2 & \theta = \beta \end{cases}, \quad \frac{3}{\Omega}\left(\psi'\sin\theta + \frac{\mu f}{\sin\theta}\right) = \begin{cases} q_1 & \theta = \alpha \\ -q_2 & \theta = \beta \end{cases} \tag{6.8}$$

Taking into account the expression for the tangential stresses (6.3) and introducing the notation $B = A + {}^1/_2\mu M$, we find

$$\tau_{\theta\varphi} = \tau - \frac{\mu}{2\sin\theta}\tau_{r\theta}, \quad \tau = \frac{B\cos\theta - C}{\sin^2\theta} \tag{6.9}$$

Using the boundary conditions (6.2), we obtain from (6.9)

$$B = \Big[q_1\sin^2\alpha + q_2\sin^2\beta + \tag{6.10}$$
$$\frac{1}{2}\mu(m_1\sin\alpha + m_2\sin\beta)\Big](\cos\alpha - \cos\beta)^{-1}$$
$$C = [q_1\sin^2\alpha\cos\beta + q_2\sin^2\beta\cos\alpha + {}^1/_2\mu(m_1\sin\alpha \times$$
$$\cos\beta + m_2\sin\beta\cos\alpha)](\cos\alpha - \cos\beta)^{-1}$$

The relationship (6.9) can also be represented in the form

$$\frac{3}{\Omega}\left(\psi'\sin\theta + \frac{\mu f}{\sin\theta}\right) + \frac{\mu}{2\Omega\sin\theta}(f' + \Phi)' = \tau \tag{6.11}$$

which, in combination with one of equations (6.6), forms a system of differential equations governing the functions $f(\theta)$, $\psi(\theta)$ and the constants A and C under the boundary conditions (6.7) and (6.8).

Introducing the new functions $\chi = f'/f$, $\Psi = \psi/f$, these differential equations can be reduced by one order. In the particular case when $m_1 = m_2 = 0$, i.e., the conic surfaces in the radial direction are ideally smooth, then by setting

$$f' + \Phi = M = 0 \tag{6.12}$$

we will have

$$B = A = \frac{q_1\sin^2\alpha + q_2\sin^2\beta}{\cos\alpha - \cos\beta}, \quad C = \frac{q_1\sin^2\alpha\cos\beta + q_2\sin^2\beta\cos\alpha}{\cos\alpha - \cos\beta}$$

$$\Omega = \frac{6\nu f'}{\sqrt{1 - \tau^2}}, \quad \nu = \operatorname{sign} f', \quad \tau_{r\theta} = \tau_{r\varphi} = u = 0 \tag{6.13}$$

$$\tau_{\theta\varphi} = \tau = \frac{q_1\sin^2\alpha(\cos\theta - \cos\beta) - q_2\sin^2\beta(\cos\alpha - \cos\theta)}{(\cos\alpha - \cos\beta)\sin^2\theta} \tag{6.14}$$

The first equation of (6.6) transforms into an identity, and from (6.11) we arrive at the differential equations

$$f'' + \left(\operatorname{ctg}\theta - \frac{2\mu\tau}{\sqrt{1 - \tau^2}}\frac{1}{\sin\theta}\right)f' - \frac{1 + \mu^2}{\sin^2\theta}f = 0 \tag{6.15}$$

which defines $f(\theta)$ under the conditions (6.7).

For normal stresses we obtain

$$\sigma_r = \sigma_\theta + \sqrt{1 - \tau^2}, \quad \sigma_\varphi = \sigma_\theta + 2\sqrt{1 - \tau^2} \tag{6.16}$$
$$\sigma_\theta = -H - A(\varphi_0 - \varphi) + 2\int_\alpha^\theta \sqrt{1 - \tau^2}\operatorname{ctg}\theta\, d\theta$$

Taking into account that the endface sections of the layer $\varphi = \pm\varphi_0$ are load-free, we obtain the condition for zero sum of the moments of the forces with respect to the axis $\theta = 0$, acting on an arbitrary part of the layer on a section of length dr between φ and φ_0

$$\int_\alpha^\beta \sigma_\varphi \sin\theta \, d\theta + (q_1 \sin\alpha + q_2 \sin\beta)(\varphi_0 - \varphi) = 0 \tag{6.17}$$

Hence, there is determined

$$H = \frac{2}{\cos\alpha - \cos\beta} \int_\alpha^\beta \sqrt{1-\tau^2} \, \frac{1 - \cos\beta\cos\theta}{\sin\theta} \, d\theta \tag{6.18}$$

The pressure, acting per unit length, on the contact surface $\theta = \alpha$ will equal

$$p_\alpha = -2r\sin\alpha \int_0^{\varphi_0} (\sigma_\theta \cos\varphi - \tau_{\theta\varphi}\sin\varphi)_{\theta=\alpha} \, d\varphi \tag{6.19}$$

from which it follows

$$p_\alpha = 2r\sin\alpha \, [H\sin\varphi_0 + (1 - \cos\varphi_0)(A + q_1)] \tag{6.20}$$

Going over to cylindrical coordinates and setting $r\sin\theta$ for $r \to \infty$ and $\theta \to 0$, we obtain from (6.14), (6.16), (6.18) and (6.19)

$$\sigma_r = -H_* - 2A_*(\varphi_0 - \varphi) + 2\int_a^r \sqrt{1-\tau^2} \, \frac{dr}{r} \tag{6.21}$$

$$\sigma_\theta = \sigma_r + 2\sqrt{1-\tau^2}, \quad \sigma_z = \sigma_r + \sqrt{1-\tau^2}$$

$$\tau_{r\theta} = \tau = A_* - \frac{B_*}{r^2}, \quad H_* = \frac{2}{b^2 - a^2} \int_a^b \sqrt{1-\tau^2}(b^2 + r^2) \frac{dr}{r}$$

$$A_* = \frac{q_1 a^2 + q_2 b^2}{b^2 - a^2}, \quad B_* = \frac{(q_1 + q_2) a^2 b^2}{b^2 - a^2}$$

$$p_a = 2a \, [H_* \sin\varphi_0 + (1 - \cos\varphi_0)(A_* + q_1)] \tag{6.22}$$

These formulas determine the ultimate state of stress of a cylindrical layer in cylindrical coordinates, when the layer is compressed between two coaxial cylindrical surfaces $u = u_1 e^{-\mu|\varphi|}$ for $r = a$, $u = -u_2 e^{-\mu|\varphi|}$ and for $r = b$. If we go over to rectangular coordinates for $r \to \infty$ and $\varphi \to 0$, the Prandtl formulas /8/ follow from (6.21), for the ultimate state of a rectangular layer during its compression by parallel rough slabs.

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